Monogenic Gaussian distribution in closed form and the Gaussian fundamental solution*

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Abstract

In this paper we present a closed formula for the CK-extension of the Gaussian distribution in $\mathbb{R}^m$, and the monogenic version of the holomorphic function $\exp(z^2/2)/z$ which is a fundamental solution of the generalized Cauchy-Riemann operator.

Keywords: Clifford analysis, Fueter's theorem, CK-extension.

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1 Introduction

The main objects of study in Clifford analysis (see e.g. [1, 3, 5, 7, 8, 18]) are the so-called monogenic functions which may be described as null solutions of the Dirac operator, the latter being the higher dimensional analogue of the Cauchy-Riemann operator.

In this paper we deal with two very well-known techniques to generate monogenic functions: the Cauchy-Kowalevski extension (CK-extension) and

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Fueter's theorem. The first technique mentioned consists in monogenically extending analytic functions in $\mathbb{R}^m$ (see e.g. [1, 3, 12, 21, 24]). The second one, named after the Swiss mathematician R. Fueter [4], gives a method to generate monogenic functions starting from a holomorphic function in the upper half of the complex plane (see [6, 9, 10, 12, 13, 14, 15, 16, 17, 19, 23]).

The aim of this paper is to illustrate how Fueter's theorem may be used to derive two special functions in Clifford analysis: the monogenic Gaussian distribution in closed form, and the Gaussian fundamental solution which generalizes the complex fundamental solution $\exp(z^2/2)/z$.

This fundamental solution plays a key role in the theory of analytic functionals with unbounded carrier (see [11]). The monogenic version is of the form

$$\mathsf{E}(x) = \frac{\overline{x}}{|x|^{m+1}} + \mathsf{M}(x), \quad x \in \mathbb{R}^{m+1} \setminus \{0\},$$

M being an entire two-sided monogenic function. It satisfies an estimate of the form

$$|\mathsf{E}(x)| \leq C \exp\left(-|\underline{x}|^2/2\right), \quad |x_0| \leq K, \quad |\underline{x}| \geq R,$$

which is crucial for a monogenic generalization of the theory of analytic functionals with carrier in a strip domain.

2 Clifford algebras and monogenic functions

Clifford algebras were introduced in 1878 by the English geometer W. K. Clifford, generalizing the complex numbers and Hamilton's quaternions (see [2]). They have important applications in geometry and theoretical physics.

We denote by $\mathbb{R}_{0,m}$ ($m \in \mathbb{N}$) the real Clifford algebra constructed over the orthonormal basis $(e_1, \ldots, e_m)$ of the Euclidean space $\mathbb{R}^m$. The basic axiom of this associative but non-commutative algebra is that the product of a vector with itself equals its squared length up to a minus sign, i.e. for any vector $\underline{x} = \sum_{j=1}^m x_j e_j$ in $\mathbb{R}^m$, we have that

$$\underline{x}^2 = -|\underline{x}|^2 = -\sum_{j=1}^m x_j^2.$$

It thus follows that the elements of the basis submit to the multiplication rules

$$\begin{aligned} e_j^2 &= -1, & j &= 1, \ldots, m, \\ e_j e_k + e_k e_j &= 0, & 1 &\leq j \neq k \leq m. \end{aligned}$$

A basis for the algebra $\mathbb{R}_{0,m}$ is then given by the elements

$$e_A = e_{j_1} \cdots e_{j_k},$$

where $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}$ is such that $j_1 < \cdots < j_k$. For the empty set $\emptyset$, we put $e_\emptyset = e_0 = 1$, the latter being the identity element. It follows that the dimension of $\mathbb{R}_{0,m}$ is 2^m.

Any Clifford number $a \in \mathbb{R}_{0,m}$ may thus be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

and its conjugate $\overline{a}$ is defined by

$$\overline{a} = \sum_A a_A \overline{e}_A, \quad \overline{e}_A = (-1)^{\frac{k(k+1)}{2}} e_A, \quad |A| = k.$$

For each $k \in \{0, 1, \ldots, m\}$, we call

$$\mathbb{R}_{0,m}^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : \ a = \sum_{|A|=k} a_A e_A \right\}$$

the subspace of k-vectors, i.e. the space spanned by the products of k different basis vectors. In particular, the 0-vectors and 1-vectors are simply called scalars and vectors respectively.

Observe that $\mathbb{R}^{m+1}$ may be naturally embedded in the Clifford algebra $\mathbb{R}_{0,m}$ by associating to any element $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ the "paravector" x given by

$$x = x_0 + \underline{x}.$$

Note that

$$\mathbb{R}_{0,m} = \bigoplus_{k=0}^{m} \mathbb{R}_{0,m}^{(k)}$$

and hence for any $a \in \mathbb{R}_{0,m}$

$$a = \sum_{k=0}^{m} [a]_k,$$

where $[a]_k$ is the projection of a on $\mathbb{R}_{0,m}^{(k)}$. By means of the conjugation, a norm $|a|$ may be defined for each $a \in \mathbb{R}_{0,m}$ by putting

$$|a|^2 = [a\overline{a}]_0 = \sum_A a_A^2.$$

Next, we introduce the Dirac operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

and the generalized Cauchy-Riemann operator

$$\partial_x = \partial_{x_0} + \partial_{\underline{x}}.$$

These operators factorize the Laplace operator in the sense that

$$\Delta_{\underline{x}} = \sum_{j=1}^{m} \partial_{x_j}^2 = -\partial_{\underline{x}}^2 \tag{1}$$

and

$$\Delta_x = \partial_{x_0}^2 + \Delta_{\underline{x}} = \partial_x \overline{\partial}_x = \overline{\partial}_x \partial_x. \tag{2}$$

Definition 1 *A function $f(\underline{x})$ (resp. $f(x)$) defined and continuously differentiable in an open set Ω of $\mathbb{R}^m$ (resp. $\mathbb{R}^{m+1}$) and taking values in $\mathbb{R}_{0,m}$, is called a (left) monogenic function in Ω if and only if it fulfills in Ω the equation*

$$\partial_{\underline{x}} f \equiv \sum_{j=1}^{m} \sum_{A} e_j e_A \partial_{x_j} f_A = 0 \quad (\textit{resp. } \partial_x f \equiv \sum_{j=0}^{m} \sum_{A} e_j e_A \partial_{x_j} f_A = 0).$$

Note that in view of the non-commutativity of $\mathbb{R}_{0,m}$ a notion of right monogenicity may be defined in a similar way by letting act the Dirac operator or the generalized Cauchy-Riemann operator from the right. Functions which are both right and left monogenic are called two-sided monogenic functions.

Two basic examples of monogenic functions are $-\underline{x}/|\underline{x}|^m$ and $\overline{x}/|x|^{m+1}$, being these functions (up to a multiplicative constant) fundamental solutions of $\partial_{\underline{x}}$ and ∂_x, respectively.

Finally, note that in view of (1) and (2) it follows that any monogenic function in Ω is harmonic in Ω and hence real-analytic in Ω.

3 Fueter's theorem

Fueter's theorem was originally formulated in the setting of quaternionic analysis (see [4]); and was later extended to the case of Clifford algebra-valued

functions by Sce [19], Qian [16] and Sommen [23]. For other generalizations we refer the reader to [6, 9, 10, 12, 13, 14, 15, 17].

Let $f(z) = u(x,y) + iv(x,y)$ $(z = x + iy)$ be a holomorphic function in some open subset $\Xi \subset \mathbb{C}^+ = \{z \in \mathbb{C} : \ y > 0\}$; and let $P_k(\underline{x})$ be a homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$, i.e.

$$\begin{aligned} \partial_{\underline{x}} P_k(\underline{x}) &= 0, & \underline{x} &\in \mathbb{R}^m, \\ P_k(t\underline{x}) &= t^k P_k(\underline{x}), & t &\in \mathbb{R}. \end{aligned}$$

Put $\underline{\omega} = \underline{x}/r$, with $r = |\underline{x}|$. In this paper, we are concerned with Sommen's generalization: *if m is an odd number, then the function*

$$\mathsf{Ft}\left[f(z), P_k(\underline{x})\right](x) = \Delta_x^{k+\frac{m-1}{2}} \left[\big(u(x_0,r) + \underline{\omega}\, v(x_0,r)\big) P_k(\underline{x})\right]$$

is monogenic in $\widetilde{\Omega} = \{x \in \mathbb{R}^{m+1} : \ (x_0, r) \in \Xi\}$.

The proof of this generalization was based on the fact that

$$\big(u(x_0,r) + \underline{\omega}\, v(x_0,r)\big) P_k(\underline{x})$$

may be written locally as $\overline{\partial}_x\big(h(x_0,r)P_k(\underline{x})\big)$ for some $\mathbb{R}$-valued harmonic function h of x_0 and r. Therefore using (2), $\mathsf{Ft}\left[f(z), P_k(\underline{x})\right]$ is monogenic if and only if (see [23])

$$\Delta_x^{k+\frac{m+1}{2}} \big(h(x_0,r)P_k(\underline{x})\big) = 0.$$

We notice that this version of Fueter's theorem provides us with the axial monogenic functions of degree k, i.e.

$$\mathsf{Ft}\left[f(z), P_k(\underline{x})\right](x) = \big(A(x_0,r) + \underline{\omega}\, B(x_0,r)\big) P_k(\underline{x}),$$

where A and B are $\mathbb{R}$-valued and continuously differentiable functions in the variables x_0 and r (see [20, 22]). It is not difficult to show that functions of this form are monogenic if and only if A and B satisfy the Vekua-type system

$$\begin{cases} \partial_{x_0} A - \partial_r B &= \dfrac{2k+m-1}{r}\, B \\ \partial_{x_0} B + \partial_r A &= 0. \end{cases} \tag{3}$$

Using this fact, we presented in [13] (see also [12]) an alternative proof which has the advantage of allowing to compute some examples. Let us give an outline of the proof. We first showed by induction that

$$A = (2k+m-1)!!\, D_r \left(k + \frac{m-1}{2}\right) \{u\},$$

$$B = (2k+m-1)!!\, D^r\left(k+\frac{m-1}{2}\right)\{v\},$$

where $D_r(n)$ and $D^r(n)$ $(n \in \mathbb{N}_0)$ are differential operators defined by

$$D_r(n)\{f\} = \left(\frac{1}{r}\,\partial_r\right)^n \{f\}, \qquad D_r(0)\{f\} = f,$$

$$D^r(n)\{f\} = \partial_r\left(\frac{D^r(n-1)\{f\}}{r}\right), \quad D^r(0)\{f\} = f.$$

Here we list some useful properties of these operators:

(i) $D^r(n)\{\partial_r f\} = \partial_r D_r(n)\{f\}$,

(ii) $D_r(n)\{\partial_r f\} - \partial_r D^r(n)\{f\} = 2n/r\, D^r(n)\{f\}$,

(iii) $D_r(n)\{fg\} = \sum_{\nu=0}^{n}\binom{n}{\nu} D_r(n-\nu)\{f\} D_r(\nu)\{g\}$,

(iv) $D^r(n)\{fg\} = \sum_{\nu=0}^{n}\binom{n}{\nu} D_r(n-\nu)\{f\} D^r(\nu)\{g\}$.

The final task was to prove that A and B satisfy the Vekua-type system (3). In order to do that, it is necessary to use the assumptions on u and v and statements (i)-(ii).

Indeed,

$$\begin{aligned}
&\partial_{x_0} A - \partial_r B\\
&\quad = (2k+m-1)!!\left(D_r\left(k+\frac{m-1}{2}\right)\{\partial_{x_0} u\} - \partial_r D^r\left(k+\frac{m-1}{2}\right)\{v\}\right)\\
&\quad = (2k+m-1)!!\left(D_r\left(k+\frac{m-1}{2}\right)\{\partial_r v\} - \partial_r D^r\left(k+\frac{m-1}{2}\right)\{v\}\right)\\
&\quad = \frac{2k+m-1}{r}\,(2k+m-1)!!\, D^r\left(k+\frac{m-1}{2}\right)\{v\}\\
&\quad = \frac{2k+m-1}{r}\, B
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{x_0} B + \partial_r A\\
&\quad = (2k+m-1)!!\left(D^r\left(k+\frac{m-1}{2}\right)\{\partial_{x_0} v\} + \partial_r D_r\left(k+\frac{m-1}{2}\right)\{u\}\right)\\
&\quad = (2k+m-1)!!\left(D^r\left(k+\frac{m-1}{2}\right)\{\partial_{x_0} v\} + D^r\left(k+\frac{m-1}{2}\right)\{\partial_r u\}\right)\\
&\quad = (2k+m-1)!!\, D^r\left(k+\frac{m-1}{2}\right)\{\partial_{x_0} v + \partial_r u\}\\
&\quad = 0,
\end{aligned}$$

which completes the proof.

It is a simple matter to check that

$$\mathsf{Ft}\left[cf(z), P_k(\underline{x})\right] = c\,\mathsf{Ft}\left[f(z), P_k(\underline{x})\right], \quad c \in \mathbb{R},$$
$$\mathsf{Ft}\left[f(z) + g(z), P_k(\underline{x})\right] = \mathsf{Ft}\left[f(z), P_k(\underline{x})\right] + \mathsf{Ft}\left[g(z), P_k(\underline{x})\right],$$

where $f(z)$ and $g(z)$ are two holomorphic functions in the upper half of the complex plane. At this point it is important to notice that $\mathsf{Ft}\left[if(z), P_k(\underline{x})\right] \neq i\mathsf{Ft}\left[f(z), P_k(\underline{x})\right]$.

It is also worth remarking that for $k = 0$, Fueter's theorem generates two-sided monogenic functions. More precisely,

$$\partial_x \mathsf{Ft}\left[f(z), 1\right] = \mathsf{Ft}\left[f(z), 1\right] \partial_x = 0 \text{ in } \widetilde{\Omega}.$$

We will now compute some examples (see also [15]).

Example 1. Let $f(z) = iz = -y + ix$. It easily follows that

$$D_r(n)\{r\} = (-1)^{n+1} \frac{(2n-3)!!}{r^{2n-1}}, \tag{4}$$

$$D^r(n)\{x_0\} = (-1)^n \frac{(2n-1)!!}{r^{2n}}\, x_0.$$

We thus get the monogenic function

$$\begin{aligned} \mathsf{Ft}\left[iz, P_k(\underline{x})\right](x) = &(-1)^{k+\frac{m-1}{2}} (2k+m-1)!!(2k+m-4)!! \\ &\times \left(\frac{1}{r^{2k+m-2}} + \frac{(2k+m-2)x_0\underline{x}}{r^{2k+m}} \right) P_k(\underline{x}), \quad \underline{x} \neq 0. \end{aligned}$$

Example 2. Consider

$$f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

It is easy to check that

$$D_r(n)\left\{ \frac{x_0}{x_0^2 + r^2} \right\} = (-1)^n \frac{2^n n! x_0}{(x_0^2 + r^2)^{n+1}}, \tag{5}$$

$$D^r(n)\left\{ \frac{r}{x_0^2 + r^2} \right\} = (-1)^n \frac{2^n n! r}{(x_0^2 + r^2)^{n+1}}. \tag{6}$$

With this choice of initial function, we obtain the well-known monogenic function in $\mathbb{R}^{m+1} \setminus \{\underline{x} \neq 0\}$:

$$\mathsf{Ft}\left[1/z, P_k(\underline{x})\right](x) = (-1)^{k+\frac{m-1}{2}}((2k+m-1)!!)^2 \left(\frac{\overline{x}}{|x|^{2k+m+1}}\right) P_k(\underline{x}).$$

Before introducing the last example, we first need to introduce the CK-extension technique.

Let $f(\underline{x})$ be an analytic function in $\mathbb{R}^m$. The CK-extension of f is the unique monogenic extension $\mathsf{CK}[f]$ of f to $\mathbb{R}^{m+1}$ and given by

$$\mathsf{CK}[f](x) = \sum_{n=0}^{\infty} \frac{(-x_0)^n}{n!} \partial_{\underline{x}}^n f(\underline{x}).$$

Example 3. Let $n \in \mathbb{N}$ and take

$$\begin{aligned} f(z) &= z^n \\ &= \sum_{\nu=0}^{[n/2]} (-1)^\nu \binom{n}{2\nu} x^{n-2\nu} y^{2\nu} + i \sum_{\nu=0}^{[(n-1)/2]} (-1)^\nu \binom{n}{2\nu+1} x^{n-(2\nu+1)} y^{2\nu+1}. \end{aligned}$$

For this initial function we have that

$$u(x_0, r) + \underline{\omega}\, v(x_0, r) = \sum_{\nu=0}^{n} \binom{n}{\nu} x_0^{n-\nu} \underline{x}^\nu.$$

Therefore $\mathsf{Ft}\left[z^n, P_k(\underline{x})\right]$ is a homogeneous monogenic polynomial of degree $n-k-m+1$ in $\mathbb{R}^{m+1}$. Moreover,

$$\mathsf{Ft}\left[z^n, P_k(\underline{x})\right](x)\Big|_{\underline{x}_0=0} = c\, \underline{x}^{n-(2k+m-1)} P_k(\underline{x}), \quad c \in \mathbb{R}.$$

From the above we can claim that $\mathsf{Ft}\left[z^n, P_k(\underline{x})\right]$ equals (up to a multiplicative constant) the CK-extension of $\underline{x}^{n-(2k+m-1)} P_k(\underline{x})$.

Proposition 1 *Let $H(z) = \sum_{n=0}^{\infty} c_n z^n$ $(c_n \in \mathbb{R})$ be an entire function. Then $\mathsf{Ft}\left[H(z), P_k(\underline{x})\right]$ is an entire monogenic function of the form*

$$\sum_{n=2k+m-1}^{\infty} C_n \mathsf{CK}[\underline{x}^{n-(2k+m-1)} P_k(\underline{x})](x), \quad C_n \in \mathbb{R}.$$

4 Monogenic Gaussian distribution

In this section, we will focus on the CK-extension of the Gaussian distribution $\exp(-|\underline{x}|^2/2)$ in $\mathbb{R}^m$. It may be given by the following series (see [3])

$$\mathsf{CK}[\exp(-|\underline{x}|^2/2)](x) = \exp(-|\underline{x}|^2/2) \sum_{n=0}^{\infty} \frac{x_0^n}{n!} H_n(\underline{x}),$$

where the functions $H_n(\underline{x})$ are polynomials in $\underline{x}$ of degree n with real coefficients and satisfy the recurrence formula

$$H_{n+1}(\underline{x}) = \underline{x} H_n(\underline{x}) - \partial_{\underline{x}} H_n(\underline{x}).$$

The polynomials $H_n(\underline{x})$ generalize the classical Hermite polynomials. It easily follows by induction that

$$\begin{aligned} H_{2n}(\underline{x}) &= \sum_{\nu=0}^{n} \binom{n}{\nu} c_n(\nu)\, \underline{x}^{2(n-\nu)}, \\ H_{2n+1}(\underline{x}) &= \sum_{\nu=0}^{n} \binom{n}{\nu} c_{n+1}(\nu)\, \underline{x}^{2(n-\nu)+1}, \end{aligned}$$

where

$$c_n(\nu) = \prod_{l=1}^{\nu} (m + 2(n-l)), \quad c_n(0) = 1.$$

From now on we will assume m an odd number. It is clear that

$$\mathsf{CK}[\exp(-|\underline{x}|^2/2)](x)\Big|_{\underline{x}=0} = \sum_{n=0}^{\infty} \frac{x_0^{2n}}{(2n)!} c_n(n).$$

This series is the Taylor expansion of the function

$$\exp(x_0^2/2) \left(1 + \sum_{n=1}^{\frac{m-1}{2}} \prod_{\nu=1}^{n} (m - (2\nu - 1)) \frac{x_0^{2n}}{(2n)!} \right).$$

Thus

$$\mathsf{CK}[\exp(-|\underline{x}|^2/2)](x)\Big|_{\underline{x}=0} = \exp(x_0^2/2) \left(1 + \sum_{n=1}^{\frac{m-1}{2}} \prod_{\nu=1}^{n} (m - (2\nu - 1)) \frac{x_0^{2n}}{(2n)!} \right).$$

Consider the holomorphic function

$$f(z) = \exp(z^2/2) = \exp\left(\frac{x^2 - y^2}{2}\right)(\cos(xy) + i\sin(xy))\,.$$

Let us now compute using Fueter's technique the corresponding monogenic function. It may be proved by induction that

$$D_r(n)\left\{\exp\left(\frac{x_0^2 - r^2}{2}\right)\right\} = (-1)^n \exp\left(\frac{x_0^2 - r^2}{2}\right), \tag{7}$$

$$D_r(n)\{\cos(x_0 r)\} = \sum_{\nu=1}^{n} a_\nu^{(n)} \frac{x_0^\nu}{r^{2n-\nu}} \cos(x_0 r + \nu\pi/2), \tag{8}$$

$$D_r(n)\{\sin(x_0 r)\} = \sum_{\nu=1}^{n} a_\nu^{(n)} \frac{x_0^\nu}{r^{2n-\nu}} \sin(x_0 r + \nu\pi/2), \tag{9}$$

$$D^r(n)\{\sin(x_0 r)\} = \sum_{\nu=0}^{n} a_{\nu+1}^{(n+1)} \frac{x_0^\nu}{r^{2n-\nu}} \sin(x_0 r + \nu\pi/2), \tag{10}$$

with

$$\begin{aligned} a_1^{(n)} &= (-1)^{n+1}(2n-3)!!, \\ a_\nu^{(n+1)} &= -(2n-\nu)a_\nu^{(n)} + a_{\nu-1}^{(n)}, \quad \nu = 2, \ldots, n, \\ a_n^{(n)} &= 1. \end{aligned}$$

By statements (iii) and (iv), we see that

$$\begin{aligned} &D_r(n)\left\{\exp\left(\frac{x_0^2 - r^2}{2}\right)\cos(x_0 r)\right\} \\ &\qquad = \exp\left(\frac{x_0^2 - r^2}{2}\right)\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^{n-\nu} D_r(\nu)\{\cos(x_0 r)\}, \end{aligned}$$

$$\begin{aligned} &D^r(n)\left\{\exp\left(\frac{x_0^2 - r^2}{2}\right)\sin(x_0 r)\right\} \\ &\qquad = \exp\left(\frac{x_0^2 - r^2}{2}\right)\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^{n-\nu} D^r(\nu)\{\sin(x_0 r)\}. \end{aligned}$$

Hence

$$\begin{aligned}
&\mathsf{Ft}\left[\exp(z^2/2), P_k(\underline{x})\right](x) = (2k+m-1)!! \\
&\times \exp\left(\frac{x_0^2 - r^2}{2}\right)\left(\sum_{\nu=0}^{k+\frac{m-1}{2}} \binom{k+\frac{m-1}{2}}{\nu}(-1)^{k+\frac{m-1}{2}-\nu} D_r(\nu)\{\cos(x_0 r)\}\right. \\
&\left. +\underline{\omega} \sum_{\nu=0}^{k+\frac{m-1}{2}} \binom{k+\frac{m-1}{2}}{\nu}(-1)^{k+\frac{m-1}{2}-\nu} D^r(\nu)\{\sin(x_0 r)\}\right) P_k(\underline{x}), \quad \underline{x} \neq 0.
\end{aligned}$$

Note that for $k=0$ the restriction of $\mathsf{Ft}\left[\exp(z^2/2), P_k(\underline{x})\right]$ to $x_0 = 0$ is

$$(-1)^{\frac{m-1}{2}}(2k+m-1)!!\exp(-|\underline{x}|^2/2).$$

Therefore, for $k = 0$, $\mathsf{Ft}\left[\exp(z^2/2), P_k(\underline{x})\right]$ equals (up to a multiplicative constant) the CK-extension of $\exp(-|\underline{x}|^2/2)$ when $\underline{x} \neq 0$.

Proposition 2 *Let m be an odd number. A closed formula for the CK-extension of the Gaussian distribution in $\mathbb{R}^m$ is given by*

$$\begin{aligned}
&\mathsf{CK}[\exp(-|\underline{x}|^2/2)](x) \\
&= \begin{cases} (-1)^{\frac{m-1}{2}} \dfrac{\mathsf{Ft}\left[\exp(z^2/2), 1\right](x)}{(m-1)!!} & \text{for } \quad \underline{x} \neq 0, \\ \exp(x_0^2/2)\left(1 + \displaystyle\sum_{n=1}^{\frac{m-1}{2}} \prod_{\nu=1}^{n} (m-(2\nu-1)) \frac{x_0^{2n}}{(2n)!}\right) & \text{for } \quad \underline{x} = 0. \end{cases}
\end{aligned}$$

For the particular case $m = 3$, we have that

$$\begin{aligned}
\mathsf{CK}[\exp(-|\underline{x}|^2/2)](x) &= \exp\left(\frac{x_0^2 - r^2}{2}\right)\left(\cos(x_0 r) + \frac{x_0}{r}\sin(x_0 r)\right. \\
&\left. + \underline{\omega}\left(\sin(x_0 r) + \frac{\sin(x_0 r)}{r^2} - \frac{x_0}{r}\cos(x_0 r)\right)\right), \text{ for } \quad \underline{x} \neq 0,
\end{aligned}$$

and

$$\mathsf{CK}[\exp(-|\underline{x}|^2/2)](x) = \exp(x_0^2/2)(1 + x_0^2), \text{ for } \quad \underline{x} = 0.$$

5 The Gaussian fundamental solution

In this last section, we will test Fueter's technique with the initial holomorphic function

$$\begin{aligned} f(z) &= \frac{\exp(z^2/2)}{z} \\ &= \frac{\exp\left(\frac{x^2-y^2}{2}\right)}{x^2+y^2}\Big(\big(x\cos(xy)+y\sin(xy)\big)+i\big(x\sin(xy)-y\cos(xy)\big)\Big). \end{aligned}$$

Note that $\exp(z^2/2)/z$ may be written as $1/z + H(z)$, where $H(z)$ is a entire function whose coefficients in the Taylor expansion around $z = 0$ are real. Thus, by Proposition 1 and Example 2, there exists an entire two-sided monogenic function M such that

$$\mathsf{Ft}\left[\exp(z^2/2)/z, 1\right](\underline{x}) = (-1)^{\frac{m-1}{2}}((m-1)!!)^2\left(\frac{\overline{x}}{|x|^{m+1}} + \mathsf{M}(x)\right).$$

We note that $\mathsf{Ft}\left[\exp(z^2/2)/z, 1\right]$ is explicitly given by

$$\begin{aligned} &\mathsf{Ft}\left[\exp(z^2/2)/z, 1\right](x) \\ &= (m-1)!!\left(D_r\left(\frac{m-1}{2}\right)\left\{\frac{\exp\left(\frac{x_0^2-r^2}{2}\right)}{x_0^2+r^2}\big(x_0\cos(x_0r)+r\sin(x_0r)\big)\right\}\right. \\ &\qquad \left. +\underline{\omega}\, D^r\left(\frac{m-1}{2}\right)\left\{\frac{\exp\left(\frac{x_0^2-r^2}{2}\right)}{x_0^2+r^2}\big(x_0\sin(x_0r)-r\cos(x_0r)\big)\right\}\right). \end{aligned}$$

In view of the above and using statements (iii) and (iv), we can also assert that $\mathsf{Ft}\left[\exp(z^2/2)/z, 1\right](x)$ may be expressed in terms of (4)-(10) with $0 \leq n \leq (m-1)/2$. Therefore

$$\mathsf{Ft}\left[\exp(z^2/2)/z, 1\right](x) = \exp\left(\frac{x_0^2-r^2}{2}\right)\big(\alpha(x_0,r) + \underline{\omega}\,\beta(x_0,r)\big),$$

where α and β are $\mathbb{R}$-valued functions.

Let $K, R > 0$. We then have that for $|x_0| \leq K$ and for $r \geq R$ the following inequality holds

$$\big|\alpha(x_0,r) + \underline{\omega}\,\beta(x_0,r)\big| \leq C,$$

where C denotes a positive constant depending on K and R.

The above observations are summarized in the proposition below.

Proposition 3 *Let m be an odd number. The function*

$$\mathsf{Ft}\left[\exp(z^2/2)/z, 1\right](x)$$

equals (up to a multiplicative constant)

$$\frac{\overline{x}}{|x|^{m+1}} + \mathsf{M}(x),$$

where $\mathsf{M}(x)$ is an entire two-sided monogenic function. Moreover, for $|x_0| \leq K$ and $r \geq R$, we have that

$$\left|\mathsf{Ft}\left[\exp(z^2/2)/z, 1\right](x)\right| \leq C \exp\left(-r^2/2\right).$$

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